can be stable at all latitudes φ, while the sufficient conditions of stability fully coincide with the Beletskii condition (4) and do not contain any parameters of the orbit of the center of mass.

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ON THE MAXIMIZATION OF THE DEGREE OF STABILITY OF A LINEAR OSCILLATING SYSTEM

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The problem of selecting optimal parameters ensuring the maximum degree of stability, is considered for the linear oscillating systems [1]. The upper bounds of the degree of stability are obtained. Necessary and sufficient conditions of attainability of the upper bound are formulated. Systems with one, two and three degrees of freedom are studied in detail. Similar problems have been already investigated in [1 – 4].

1. Statement of the problem. We consider a system the motion of which is described by the following linear differential equation:

$$Ax^{\cdot\cdot} + Bx^{\cdot} + Cx = 0 \tag{1.1}$$

Here x is an n-dimensional vector, A, B and C are $n \times n$ matrices and a dot denotes the derivative with respect to time. Equation (1.1) can describe e.g. small oscillations of a mechanical system about the position of equilibrium $x = 0$. Problems of the stability of the system of the type (1.1) were studied in detail in [5]. Let us consider two problems concerning the choice of the parameters of the system.

Problem 1. Let A and C be given positive definite matrices. We require to find a real matrix B such that the system (1.1) is stable and its degree of stability is the highest possible (the degree of stability of a stable system is given by the quantity $\min_{1\leqslant j\leqslant 2n} |\mathrm{Re}\,\lambda_j^\circ|$ [1]). Here λ_j° $(j = 1, 2, \ldots, 2n)$ denote the roots of the characteristic polynomial of the system (1.1).

Problem 2. Let A and B be given positive definite matrices. We require to find a real matrix C such that the system (1.1) is stable and its degree of stability is the highest possible.

In Problems 1 and 2 the matrix A, as stipulated, is positive definite and therefore, nonsingular, and the system (1.1) is equivalent to the system

$$x^{\cdot\cdot} + A^{-1}Bx^{\cdot} + A^{-1}Cx = 0 \tag{1.2}$$

2. Basic results. Expanding the characteristic determinant, we obtain the characteristic polynomial of the system (1.2) in the form

$$\Delta(\lambda) = \lambda^{2n} + \mathrm{tr}\,(A^{-1}B)\,\lambda^{2n-1} + a_{2n-2}(b_{ij}, c_{ij})\,\lambda^{2n-2} + \ldots \tag{2.1}$$
$$a_1(b_{ij}, c_{ij})\,\lambda + R_1, \qquad R_1 = \det C / \det A$$

The expression $a_k(b_{ij}, c_{ij})$ $(k = 1, 2, \ldots, 2n-2)$ means that the coefficients of λ^k are functions of the elements of the matrices B and C, and the symbol tr denotes the trace of the matrix.

Using the relations

$$R_1 = \prod_{j=1}^{2n} \lambda_j^\circ, \qquad \mathrm{tr}\,(A^{-1}B) = -\sum_{j=1}^{2n} \mathrm{Re}\,\lambda_j^\circ$$

and the fact that the real parts of the roots of the characteristic polynomial of a stable system are nonpositive, we can prove the following assertions.

Assertion 1. If A and C are given positive definite matrices and the system (1.2) is stable, then

$$\min_{1\leqslant j\leqslant 2n} |\mathrm{Re}\,\lambda_j^\circ| \leqslant R_{2n}, \qquad R_{2n} = \sqrt[2n]{\det C / \det A}$$

and the equality is attained if and only if all roots of the characteristic polynomial are equal to $\lambda_j^\circ = -R_{2n}$, $j = 1, 2, \ldots, 2n$.

Assertion 2. If A and B are given positive definite matrices and the system (1.2) is stable, then

$$\min_{1\leqslant j\leqslant 2n} |\mathrm{Re}\,\lambda_j^\circ| \leqslant \mathrm{tr}\,(A^{-1}B) / 2n$$

and the equality is attained if and only if

$$\mathrm{Re}\,\lambda_j^\circ = -\,\mathrm{tr}\,(A^{-1}B) / 2n, \qquad j = 1, 2, \ldots, 2n$$

i.e. when all roots of the characteristic polynomial have the form

$$\lambda^\circ_{2s-1,2s} = -\,\mathrm{tr}\,(A^{-1}B) / 2n \pm i\Omega, \qquad s = 1, 2, \ldots, n$$

From Assertion 1 it follows that a system with a degree of stability equal to R_{2n} has a characteristic polynomial equal to $\Delta(\lambda) = (\lambda + R_{2n})^{2n}$, and its coefficients are

$$a_k = C_{2n}^k R_{2n}^{2n-k}, \qquad k = 0, 1, \ldots, 2n \tag{2.2}$$

where C_{2n}^k are the binomial coefficients. If the matrices A and C are specified, then the coeificients of the characteristic polynomial (2.1) are, with the exception of the free term and of the coefficient of λ^{2n} which is equal to unity, definite functions of the elements of the matrix B. Equating the coefficients of the polynomial (2.1) dependent on the elements of B with the corresponding coefficients of (2.2), we obtain a system of $2n - 1$ equations (since only $2n - 1$ coefficients of the characteristic polynomial depend on the elements of B) for n^2 unknown elements of this matrix

$$\operatorname{tr}(A^{-1}B) = C_{2n}^{2n-1} R_{2n}, \qquad (2.3)$$
$$a_l(b_{ij}, c_{ij}) = C_{2n}^l R_{2n}^{2n-l}, \qquad l = 1, 2, \ldots, 2n-2$$

The degree of stability of the system (1.2) attains its upper limit equal to R_{2n} in the class of real matrices B (with the positive definite matrices A and C given), if and only if the system of equations (2.3) has at least one real solution relative to b_{ij} $(i, j = 1, 2, \ldots, n)$. Any real matrix the elements of which satisfy the system of equations (2.3), is the optimal matrix dor Problem 1.

From Assertion 2 it follows that the characteristic polynomial of a system with the degree of stability $\operatorname{tr}(A^{-1}B) / 2n$ is equal to

$$\Delta(\lambda) = \prod_{s=1}^{n} [(\lambda + \operatorname{tr}(A^{-1}B) / 2n)^2 + \Omega_s^2] \qquad (2.4)$$

The last expression enables us to express the coefficients of the polynomial as functions of Ω_s $(s = 1, 2, \ldots, n)$. Let us denote the coefficients of λ^k expressed as functions of Ω_s $(s = 1, 2, \ldots, n)$, by $a_k(\Omega_s)$ $(k = 0, 1, 2, \ldots, 2n-2)$. With A and B given, the coefficients $a_k(b_{ij}, c_{ij})$ $(k = 1, 2, \ldots, 2n-2)$ and the free term R_1 of the characteristic polynomial (2.1), are all definite functions of the elements of the matrix C. Equathing the coefficients of the characteristic polynomial dependent on the elements of C with the corresponding coeffcients $a_k(\Omega_s)$, we obtain the following system of $2n - 1$ equations for n^2 unknown elements of this matrix:

$$a_l(b_{ij}, c_{ij}) = a_l(\Omega_s), \quad l = 1, 2, \ldots, 2n-2, \quad R_1 = a_0(\Omega_s) \qquad (2.5)$$

The degree of stability of the system (1.2) attains its upper limit equal to $\operatorname{tr}(A^{-1}B) / 2n$ in the class of real matrices C (with the positive definite matrices A and B given), if and only if the system (2.5) has at least one real solution relative to c_{ij} $(i, j = 1, 2, \ldots, n)$. Any real matrix the elements of which satisfy the system (2.5), is the optimal matrix of Problem 2. We note that the system (2.5) has n additional arbitrary real parameters Ω_s $(s = 1, 2, \ldots, n)$. The arbitrariness can be utilized for ensuring that certain conditions are met, e.g. for making the matrix C positive definite.

3. Systems with $n \leqslant 3$. Let us consider one-, two- and three-dimensional systems. Solving Problem 1, we assume that $A = E$ is a unit matrix and C is a diagonal matrix

$$C = \operatorname{diag}(\omega_1^2, \omega_2^2, \ldots, \omega_n^2), \qquad \omega_i^2 \neq 0, \ i = 1, 2, \ldots, n$$

In solving Problem 2 we assume that $A = E$ is a unit matrix and B is a diagonal matrix with positive diagonal elements

$$B = \operatorname{diag}(b_{11}, b_{22}, \ldots, b_{nn})$$

The above assumptions do not lead to any loss of generality, since the matrices A and C in Problem 1 and A and B in Problem 2, as stipulated, are positive definite ones. Consequently a single nonsingular transformation is sufficient to convert A into a unit matrix and C (in Problem 1) or B (in Problem 2) into a diagonal matrix with positive diagonal elements.

1). $n = 1$. In this case the characteristic polynomial has the form $\Delta(\lambda) = \lambda^2 + b\lambda + \omega^2$.

Solution of Problem 1. In accordance with the results stated in Sect. 2, the optimal value of b is equal to 2ω and the corresponding highest degree of stability is ω. The optimal parameter can, in this case, be determined uniquely.

Note 1. When $\omega_1^2 = \omega_2^2 = \ldots = \omega_n^2 = \omega^2$ and n is arbitrary, the diagonal matrix $B = \mathrm{diag}(2\omega, 2\omega, \ldots, 2\omega)$ is the optimal matrix for Problem 1, and the corresponding degree of stability is ω.

Solution of Problem 2. When $n = 1$, Eq. (2.4) has the form

$$\Delta(\lambda) = \lambda^2 + b\lambda + \Omega^2 + b^2/4$$

The system (2.5) reduces to a single equation $c = b^2/4 + \Omega^2$, the optimal value of $c = b^2/4 + \Omega^2$ and the corresponding degree of stability is $b/2$. The value of the optimal coefficient is determined in this case not exactly as in Problem 1, but to within an arbitrary positive term.

Note 2. When $b_{11} = b_{22} = \ldots = b_{nn} = b$ and n is arbitrary, the diagonal matrix

$$C = \mathrm{diag}\left(\frac{b^2}{4} + \Omega_1^2, \frac{b^2}{4} + \Omega_2^2, \ldots, \frac{b^2}{4} + \Omega_n^2\right)$$

is the optimal matrix for Problem 2, and the corresponding degree of stability is $b/2$.

2). $n = 2$. Solution of Problem 1. In this case the system (2.3) has the form

$$b_{11} + b_{22} = 4\sqrt{\omega_1\omega_2} \tag{3.1}$$

$$\omega_1^2 + \omega_2^2 + b_{11}b_{22} - b_{12}b_{21} = 6\omega_1\omega_2 \tag{3.2}$$

$$\omega_2^2 b_{11} + \omega_1^2 b_{22} = 4(\omega_1\omega_2)^{3/2} \tag{3.3}$$

If $\omega_1^2 \neq \omega_2^2$, then b_{11} and b_{22} are determined uniquely from (3.1) and (3.3), and the product $b_{12}b_{21}$ is found uniquely from (3.2)

$$b_{11} = \frac{4\omega_1\sqrt{\omega_1\omega_2}}{\omega_1 + \omega_2}, \quad b_{22} = \frac{4\omega_2\sqrt{\omega_1\omega_2}}{\omega_1 + \omega_2} \tag{3.4}$$

$$b_{12}b_{21} = (\omega_1 - \omega_2)^4/(\omega_1 + \omega_2)^2$$

Thus for $\omega_1^2 \neq \omega_2^2$ the optimal matrix is determined by Eqs. (3.1) – (3.3) to within the product of the off-diagonal matrix elements. Since the product $b_{12}b_{21}$ is positive, it is possible to select matrix B as the symmetric matrix which maximizes the degree of stability. If symmetry is stipulated, the optimal matrix is determined to within the sign of the off-diagonal matrix elements. If $\omega_1^2 = \omega_2^2$, then (3.1) and (3.3) are linearly dependent and the elements b_{11}, b_{22} can be selected in an infinite number of ways. We note that b_{11} and b_{22} determined by (3.4) represent a solution of the system (3.1), (3.3) also when $\omega_1^2 = \omega_2^2$. In this case $b_{12}b_{21} = 0$ and the matrix can be chosen diagonal.

Let us establish the conditions of positive definiteness of the optimal matrix. The necessary and sufficient condition of positive definiteness is that $b_{11} > 0$ and $\det B > 0$. The first inequality follows from (3.4), hence the optimal matrix is positive definite if and only if

$$\det B = 6\omega_1\omega_2 - \omega_1^2 - \omega_2^2 > 0$$

This inequality in turn holds if and only if

$$3 - 2\sqrt{2} < \omega_1/\omega_2 < 3 + 2\sqrt{2} \tag{3.5}$$

In this case the upper limit of the degree of stability can be secured by the dissipative forces.

Solution of Problem 2. In the two-dimensional case the system of equations (2.5) has the form

$$c_{11} + c_{22} + b_{11}b_{22} = {}^3/_8\,(b_{11} + b_{22})^2 + \Omega_1^2 + \Omega_2^2 \tag{3.6}$$

$$c_{11}b_{22} + c_{22}b_{11} = {}^1/_2\,(\Omega_1^2 + \Omega_2^2)\,(b_{11} + b_{22}) + {}^1/_{16}\,(b_{11} + b_{22})^3 \tag{3.7}$$

$$c_{11}c_{22} - c_{12}c_{21} = {}^1/_{256}\,(b_{11} + b_{22})^4 + {}^1/_{16}\,(b_{11} + b_{22})^2(\Omega_1^2 + \Omega_2^2) + \Omega_1^2\Omega_2^2 \tag{3.8}$$

If $b_{11} \neq b_{22}$, then the system (3.6), (3.7), with Ω_1 and Ω_2 given, has a unique solution relative to c_{11} and c_{22}

$$c_{11} = {}^1/_2\,(\Omega_1^2 \mp \Omega_2^2) + {}^1/_{16}\,(b_{11} - b_{22})^2 + {}^1/_4\,b_{11}^2$$
$$c_{22} = {}^1/_2\,(\Omega_1^2 + \Omega_2^2) + {}^1/_{16}\,(b_{11} - b_{22})^2 + {}^1/_4 b_{22}^2$$

Equation (3.8) yields the product $c_{12}c_{21}$

$$c_{12}c_{21} = {}^1/_{64}\,(b_{22} - b_{11})^4 + {}^1/_8\,(b_{22} - b_{11})^2\,(\Omega_1^2 + \Omega_2^2) + {}^1/_4\,(\Omega_1^2 - \Omega_2^2)^2$$

which is positive for any Ω_1 and Ω_2, and it follows that the optimal matrix C can be chosen symmetric.

We note that the matrix C can always be chosen positive definite. In fact, for the positive definiteness it is necessary and sufficient that the matrix be symmetric and that the following inequalities hold:

$$c_{11} > 0, \qquad c_{11}c_{22} - c_{12}c_{21} > 0$$

The above inequalities follow directly from the expression for c_{11} and from (3.8), respectively.

If $b_{11} = b_{22}$, then Eqs. (3.6) and (3.7) are linearly dependent and the diagonal elements can be chosen in an infinite number of ways.

Thus, in the two-dimensional case the upper limit of the degree of stability is equal to $\sqrt{\omega_1\omega_2}$ in the case of Problem 1, and to $(b_{11} + b_{22})\,/\,4$ in the case of Problem 2.

3) $n = 3$. In the three-dimensional case we shall limit ourselves to the proof of existence of a matrix which provides the upper limit of the dagree of stability.

Solution of Problem 1. In the three-dimensional case Eqs. (2.3) have the form

$$b_{11} + b_{22} + b_{33} = 6\sqrt[6]{\omega_1^2\omega_2^2\omega_3^2} \tag{3.9}$$

$$\omega_1^2 + \omega_2^2 + \omega_3^2 + b_{11}b_{22} + b_{11}b_{33} + b_{23}b_{33} - b_{23}b_{32} - b_{12}b_{21} - b_{13}b_{31} = \quad (3.10)$$
$$15\sqrt[6]{\omega_1^4\omega_2^4\omega_3^4} \tag{3.11}$$
$$(\omega_2^2 + \omega_3^2)\,b_{11} + (\omega_1^2 + \omega_3^2)b_{22} + (\omega_1^2 + \omega_2^2)\,b_{33} +$$
$$b_{11}b_{22}b_{33} - b_{11}b_{23}b_{32} - b_{12}b_{21}b_{33} + b_{12}b_{31}b_{23} + b_{13}b_{21}b_{32} - b_{13}b_{31}b_{22} =$$
$$20\omega_1\omega_2\omega_3$$

$$\omega_1^2\omega_2^2 + \omega_1^2\omega_3^2 + \omega_2^2\omega_3^2 + \omega_2^2b_{11}b_{33} + \omega_3^2b_{11}b_{22} + \omega_1^2b_{22}b_{33} - \quad (3.12)$$
$$\omega_1^2b_{23}b_{32} - \omega_3^2b_{12}b_{21} - \omega_2^2b_{13}b_{31} = 15\sqrt[6]{\omega_1^8\omega_2^8\omega_3^8}$$

$$\omega_2^2\omega_3^2b_{11} + \omega_1^2\omega_3^2b_{22} + \omega_1^2\omega_2^2b_{33} = 6\sqrt[6]{\omega_1^{10}\omega_2^{10}\omega_3^{10}} \tag{3.13}$$

If $\omega_1^2 = \omega_2^2 = \omega_3^2$, then the solution exists (see Note 1). Let us consider the case when $\omega_1^2 \neq \omega_2^2$ (similarly we can consider the cases $\omega_1^2 \neq \omega_3^2$ and $\omega_2^2 \neq \omega_3^2$). Equations (3.9) and (3.13) representing a system of two linear equations for three unknown diagonal elements of the matrix B have compatible solutions if $\omega_1^2 \neq \omega_2^2$, since in this case the rank of the matrix of the coefficients accompanying the unknowns, is two. Equations (3.10) and (3.12) representing a system of two linear equations for three unknown products $b_{12}b_{21}$, $b_{13}b_{31}$ and $b_{23}b_{32}$ have compatible solutions for any diagonal elements of the matrix B and any value of the product $b_{12}b_{21}$. Let us set $b_{12} = 0$. Then Eq. (3.11) has a solution relative to b_{21} since one can always make $b_{13}b_{32} \neq 0$. This proves that a solution exists, therefore the upper limit of the degree of stability which is equal to $(\omega_1\omega_2\omega_3)^{1/3}$, can be attained.

In the case of Problem 2, the attainability of the upper limit can be proved in the same manner.

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DYNAMICS OF UNWINDING A FILAMENT

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The motion of a heavy flexible filament being unwound from a rotor is investigated. The aerodynamic drag is taken into account. The possibility is shown of realizing a steady-state process and its investigation is given.

Rapidly-rotating rotors are often fabricated by means of multilayer filament windings [1]. When one of the peripheral turns is ruptured, the effect of aerodynamic drag can prevent complete unwinding of the filament. It is of interest to investigate the possibility of a stationary rotational process for an incompletely unwound filament in the case of a constant angular rotor velocity and the effect of aerodynamic drag, and also to determine the shape and tension of the free part of the filament (not lying on the rotor), the limit radius of the unwinding and the force of interaction between this part of the